Analysis Problem

Each test counts towards one half of the grade.

Notations

- $\mathbb{Z}, \mathbb{R}$ denotes respectively the sets of relative integers, and real numbers; $\mathbb{N}$ denotes the set of non negative integers, and $\mathbb{N}^*$ the set of positive integers.
- $[0, 1)$ denotes the set of real numbers $x \in \mathbb{R}$ such that $0 \leq x < 1$. We let $C = [0, 1) \times [0, 1)$.
- Any vector $v \in \mathbb{R}^2$ writes uniquely $v = [v] + \{v\}$, where $[v]$ is an element of $\mathbb{Z}^2$ and $\{v\} \in C$; $[v]$ is called the **integer part** of v and $\{v\}$ the **fractional part** of v.
- $A \in M_2(\mathbb{Z})$ denotes a 2×2 matrix with integer entries.
- $T_A : C \to C$ is the map defined by $T_A(v) = \{Av\}$.
- For every $n \in \mathbb{N}$, we define T_A^n by induction: $T_A^0 = I$ and $T_A^n = T_A \circ T_A^{n-1}$ for $n > 0$.
- A subset $E \subset C$ is **dense** in C if any euclidean disc $D \subset C$ of positive radius contains a point of E.

Part I

I.1. Prove that if $A, B \in M_2(\mathbb{Z})$, then we have $T_{AB} = T_A \circ T_B$.
I.2. A **euclidean segment** of C is the intersection of C with a segment contained in an affine line of $\mathbb{R}^2$. Prove that if $\det(A) = 0$, then the image of T_A is a finite union of euclidean segments.
I.3. A point $v \in C$ is called T_A**-periodic** if there exists $n \in \mathbb{N}^*$ such that $T_A^n v = v$. Prove that if $\det(A) \neq 0$, the set of periodic points of T_A is dense in C. Hint: Observe that for p a prime number not dividing $\det(A)$, the map T_A induces a bijection from $C \cap \frac{1}{p}\mathbb{Z}^2$ to itself.

Part II

In this part, we let

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

The T_A**-orbit** *of a point* $v \in C$ *is the set of points of the form* $T_A^n(v)$*, where* $n \in \mathbb{N}$*. The goal of this part is to prove that there exists a point* $v \in C$ *whose* T_A*-orbit is dense in* C*.*

II.1. Show that for every $w \in \mathbb{Z}^2$ different from the origin, $A^n(w)$ tends to infinity when n tends to infinity. Hint: Compute the eigenvalues of A and observe that the eigenvectors of A are irrational.

II.2. Let $f, g : \mathbb{R}^2 \to \mathbb{C}$ be continuous functions which are $\mathbb{Z}^2$ periodic, namely $f(v + w) = f(v)$ and $g(v + w) = g(v)$ for every $v \in \mathbb{R}^2$ and every $w \in \mathbb{Z}^2$. Show that f and g can be uniformly approximated by trigonometric polynomials of the form

$$P(v) = \sum_{w \in \mathbb{Z}^2} a_w \exp(2i\pi \; v \cdot w)$$

where a_w are complex numbers all of whose vanish but a finite number, and where $v \cdot w$ denotes the usual scalar product on $\mathbb{R}^2$.

II.3. Deduce from II.1 and II.2 that the multiple integrals

$$\int_0^1 \int_0^1 f(x, y) g(A^n(x, y)) dx dy$$

converge when n tends to infinity to the product

$$\left(\int_0^1 \int_0^1 f(x, y) dx dy \right) \cdot \left(\int_0^1 \int_0^1 g(x, y) dx dy \right).$$

II.4. Prove that for every euclidean disc of positive radius $D \subset C$, the set $\bigcup_{n \in \mathbb{N}} T_A^{-n}(D)$ is dense in C.

II.5. Prove that there exists a sequence $\{D_k\}_{k \in \mathbb{N}}$ of euclidean discs $D_k \subset C$ of positive radius such that every euclidean disc $D \subset C$ of positive radius contains one of the D_k's.

II.6. Prove by induction that there is a sequence $\{E_k\}_{k \in \mathbb{N}}$ of closed euclidean discs $E_k \subset C$ of positive radius such that $E_{k+1} \subset E_k$ and such that for every k, there exists an integer n_k such that $T_A^{n_k}(E_k) \subset D_k$.

II.7. Conclude.